Fourier Algorithms

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Outline

Full Cycle Fourier Algorithm

- **Data window length** characterised by:
  1. Number of sample points
  2. Time span of the window
- For example, a 3-sample data window spans $2\Delta t$.
- **Full cycle Fourier**: Data window spans one cycle subject to $N > 2$

- With K samples per cycle, the *least-squares* model:

$$\begin{bmatrix} \sin\theta_{K-1} & \cos\theta_{K-1} \\ \sin\theta_{K-2} & \cos\theta_{K-2} \\ | & | \\ | & | \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} V_m\cos\phi_v \\ V_m\sin\phi_v \end{bmatrix} + \begin{bmatrix} e_{K-1} \\ e_{K-2} \\ | \\ | \\ e_0 \end{bmatrix} = \begin{bmatrix} v_{K-1} \\ v_{K-2} \\ | \\ | \\ v_0 \end{bmatrix}$$

- The LS solution:

$$\begin{bmatrix} \sum_{j=0}^{K-1} \sin^2\theta_j & \sum_{j=0}^{K-1} \sin\theta_j\cos\theta_j \\ \sum_{j=0}^{K-1} \sin\theta_j\cos\theta_j & \sum_{j=0}^{K-1} \cos^2\theta_j \end{bmatrix} \begin{bmatrix} V_m\cos\phi_v \\ V_m\sin\phi_v \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{K-1} \sin\theta_j . v_j \\ \sum_{j=0}^{K-1} \cos\theta_j . v_j \end{bmatrix}$$

Show that,

$$\begin{bmatrix} \frac{K}{2} & 0 \\ 0 & \frac{K}{2} \end{bmatrix} \begin{bmatrix} V_m \cos\phi_v \\ V_m \sin\phi_v \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{K-1} v_j \sin\theta_j \\ \sum_{j=0}^{K-1} v_j \cos\theta_j \end{bmatrix} \tag{1}$$

$$\implies \begin{cases} V_m \cos\phi_v = \frac{2}{K} \sum_{j=0}^{K-1} v_j \sin\theta_j \\ V_m \sin\phi_v = \frac{2}{K} \sum_{j=0}^{K-1} v_j \cos\theta_j \end{cases} \tag{2}$$

where, $\theta_j = \frac{2\pi}{K} j$

Hint:

$$\sum_{j=0}^{K-1} \sin^2\theta_j \Big/ \sum_{j=0}^{K-1} \cos^2\theta_j \equiv \int_0^{2\pi/\omega_0} \sin^2\omega_0 t dt \Big/ \int_0^{2\pi/\omega_0} \cos^2\omega_0 t dt = \frac{K}{2}$$

$$\sum_{j=0}^{K-1} \sin\theta_j \cos\theta_j \equiv \int_0^{2\pi/\omega_0} \sin 2\omega_0 t dt = 0$$

- The voltage signal $v(t) = V_m \sin(\omega t + \phi_v)$ is also represented in literature as

$$V_m \cos\phi_v \sin\omega t + V_m \sin\phi_v \cos\omega t = V_s \sin\omega t + V_c \cos\omega t$$

- With this notation, $V_s = V_m \cos\phi_v$ and $V_c = V_m \sin\phi_v$.

- The estimation equations in (2) can be generalised for the L^{th} window as,

$$V_s^L = \frac{2}{K} \sum_{j=L-K+1}^{L} v_j \sin \theta_j \tag{3}$$

$$V_c^L = \frac{2}{K} \sum_{j=L-K+1}^{L} v_j \cos \theta_j \tag{4}$$

- **Convention:** latest sample corresponds to the window number
- These equations are identical to the DFT equations.

Randn Multi-plier (E)	2-point algorithm		3-point algorithm		Full cycle Fourier algorithm (K=10)	
	μ	σ	μ	σ	μ	σ
0.1	10.0069	0.1596	10.0061	0.0927	10.0058	0.0441
0.5	10.0596	0.7991	10.0387	0.4641	10.0308	0.2205
1.0	10.1824	1.5938	10.0982	0.9287	10.0656	0.4409
1.5	10.3707	2.3683	10.1780	1.3927	10.1045	0.6610
2.0	10.6346	3.0919	10.2781	1.8547	10.1475	0.8806
2.5	10.9825	3.7529	10.3985	2.3130	10.1945	1.0995
3.0	11.4055	4.3830	10.5400	2.7638	10.2455	1.3176

Inferences

- *accuracy* versus *speed*
- σ for full Fourier $\Rightarrow$ improved accuracy
- 2, 3 sample algorithms $\Rightarrow$ faster performance

Example 1: Harmonic and Noise Filtering Capability of the Full Cycle Algorithm

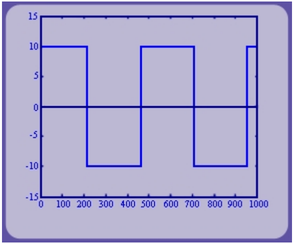


Figure: Square Periodic Wave

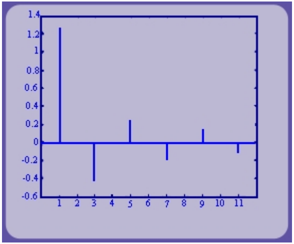


Figure: Harmonic Spectrum

- Input: 50 Hz square wave plus random noise
- Sampling rate: 10 samples per cycle
- True value of the fundamental: $\frac{4}{\pi} \times 10 = 12.7324$

Illustration of the Harmonic and Noise Filtering Capability

Randn multiplier(E)	Mean	Standard deviation
0.1	12.9512	0.0444
0.5	12.9802	0.2220
1.0	13.0193	0.4440
1.5	13.0614	0.6660
2.0	13.1065	0.8879
2.5	13.1546	1.1096
3.0	13.2057	1.3310

- *Note:* μ and σ are calculated over 100 estimations.

Example 2: Half Cycle Fourier Algorithm

- Window length: Half cycle $\Rightarrow$ Faster Estimation
- Estimation equations with K (even) samples per half cycle:

$$V_c^L = \frac{2}{K} \sum_{j=L-K+1}^{L} v_j \cos \theta_j \tag{5}$$

$$V_s^L = \frac{2}{K} \sum_{j=L-K+1}^{L} v_j \sin \theta_j \tag{6}$$

- First K windows are incomplete $\Rightarrow$ zeroes are padded at the beginning
- Correct results available only after $L \geq K$.

The table in the next slide compares the performance of the half cycle Fourier algorithm with the 2- and 3-point algorithms

Randn Multi-plier (E)	2-point algorithm		3-point algorithm		Full cycle Fourier algorithm (K=10)	
	μ	σ	μ	σ	μ	σ
0.1	10.0069	0.1596	10.0061	0.0927	10.0058	0.0614
0.5	10.0596	0.7991	10.0387	0.4641	10.0322	0.3070
1.0	10.1824	1.5938	10.0982	0.9287	10.0727	0.6138
1.5	10.3707	2.3683	10.1780	1.3927	10.1214	0.9201
2.0	10.6346	3.0919	10.2781	1.8547	10.1783	1.2254
2.5	10.9825	3.7529	10.3985	2.3130	10.2434	1.5294
3.0	11.4055	4.3830	10.5400	2.7638	10.3168	1.8314

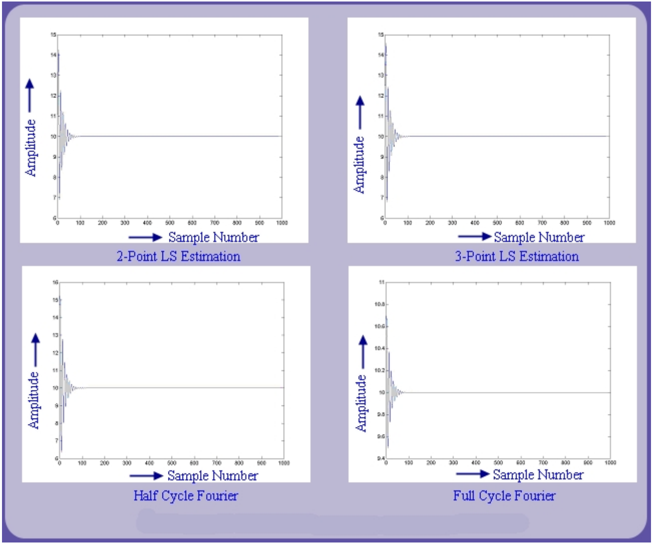
Example 3: Comparison of DC Filtering by the Estimation Algorithms

- Consider,

$$i(t) = 10\sin(2\pi \times 50 \times t - 30^\circ) + 5e^{-t\times 2\pi \times \frac{50}{10}}$$

- The figure in the following slide shows the estimated magnitude of I_m, measured for 5-fundamental cycles, using the 2-point, 3-point, half-cycle and full-cycle Fourier algorithms.

Comparison of DC Filtering by the Estimation Algorithms

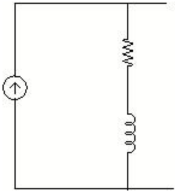


Remarks

- Significant errors are present in all methods.
- The full-cycle Fourier algorithm is the most accurate.
- DC offset current $\Leftrightarrow$ noise $\Rightarrow$ Non-zero mean
  Thus, least-square estimation algorithms are expected to fail.
- Use some other filter for the DC offset current: *mimic impedance*.

Mimic Impedance

- An impedance whose $\frac{X}{R}$ ratio is identical $\frac{X}{R}$ ratio of transmission lines.



The sinusoidal voltage developed across the mimic impedance is given by

$$
\begin{aligned}
v(t) &= R_1 i + L_1 \frac{di}{dt} \\
&= R_1 I \sin(\omega t - \phi) + \omega L_1 I \cos(\omega t - \phi) + R_1 I_0 e^{\frac{-t}{\tau}} - \frac{L}{\tau} I_0 e^{\frac{-t}{\tau}} \\
&= Z_1 I \sin(\omega t - \phi + \theta) + I_0 e^{\frac{-t}{\tau}} [R_1 - \frac{L_1}{\tau}] \\
&= Z_1 I \sin(\omega t - \phi + \theta) + L_1 I_0 e^{\frac{-t}{\tau}} [\frac{R_1}{L_1} - \frac{1}{\tau}]
\end{aligned}
$$

where time constant τ is the L/R ratio of the line.

Remarks

- If $\dfrac{L_1}{R_1} = \tau \Rightarrow$ No DC offset component and the voltage is:

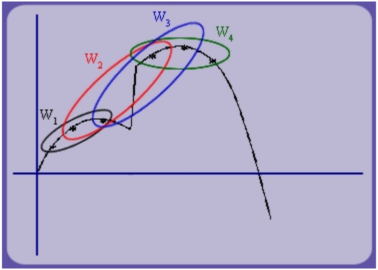
$$v(t) = Z_1 I \sin(\omega t - \phi + \theta)$$

- This is the sinusoidal-steady response.
- Current: scaled in magnitude and out of phase.
- Mimic impedance output: sinusoidal current

Effect of data window length

Illustration for a 3 sample window.

- W1→Pre-fault data⇒Correct estimate
- W2→Post- and Pre-fault samples⇒Erroneous estimate
- W4→Post-fault samples alone⇒Correct estimate

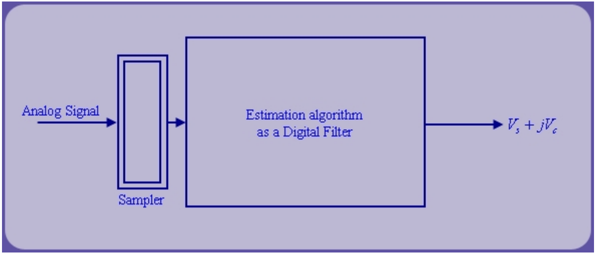


Remarks

- Delay introduced in measuring post-fault signal is equal to length of the data window.
- Thus, CT may be driven into saturation by the DC offset current.
- Half-cycle window reduces accuracy of estimation but there may be no CT saturation.

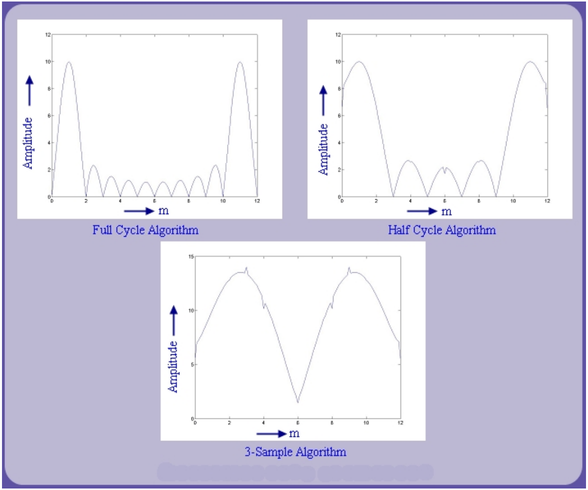
Frequency Response of Estimation Algorithms

- Estimation algorithms can be viewed as digital filters to extract the fundamental.
- Harmonics Filtering $\Leftrightarrow$ Frequency response of the estimation algorithm



- Filter input: samples at mf_0, $m = (0, \pm 1, \pm 2, \cdots)$
- Filter output: fundamental component
- $m = \pm 1 \Rightarrow$ output follows input
- $m \neq 1 \Rightarrow$ output should be zero

The frequency response for the 3-sample, half-cycle and full-cycle algorithms are shown in the following slide.


Amplitude
m
Full Cycle Algorithm
Amplitude
m
Half Cycle Algorithm
Amplitude
m
3-Sample Algorithm

Remarks

1. Full-cycle algorithm: rejects DC component and all harmonics efficiently
2. Half-cycle algorithm: rejects odd but not even harmonics efficiently
3. 3-sample algorithm: poor harmonic rejection
4. *Acharacteristic* frequencies are wrongly interpreted by all algorithms as fundamental.

Review Questions

Exercise 1

Consider evaluation of $\int\limits_0^{2\pi/\omega_0} \sin 2\omega_0 t dt$ by the trapezoidal rule of integration. This is the average of the second harmonic signal over 2-cycles which is known to be zero. Consider sampling this signal at the rate of K-samples per cycle corresponding to the fundamental frequency. The samples are at $t = 0, \cdots, \frac{2\pi}{K}(2K - 1)$. Now append the $K + 1$ sample at the end. Clearly, $\sin(2\bar{K}\theta) = 0$ and $\theta = \frac{2\pi}{K}$. Addition of this sample allows us to cover one full cycle length of fundamental on x-axis.

Now, show that $\sum_{j=0}^{k-1} \sin \frac{2\pi}{K} j$ is the numerical evaluation of this integral. Hence, deduce that $\sum_{j=0}^{k-1} \sin \frac{2\pi}{K} j = 0$. Illustrate your result geometrically.

Exercise 2

Assuming a sampling rate of 32 samples per cycle, generate samples for a 50 Hz sinusoidal signal with $V_m = 10$ at different levels of noise. Now, choose noise parameter choose $E = 0.5$. Consider the standard deviation of the estimations obtained after 100 trials. Plot the curves of σ vs K (the no. of cycles in the data window), where K is varied from 1 to 4. Hence, show that increasing the length of the data window reduces the estimation error. Interpret this result in terms of speed vs accuracy conflict in relaying.

Exercise 3
Repeat exercise 2 for $E = 0.1, 1, 2, 3$ and 4.
Exercise 4
Consider LS estimate of phasor using half cycle data window i.e. K-samples per half cycle at nominal frequency. Show that the estimate equations are given as below:

$$\begin{bmatrix} \sum_{j=0}^{K-1} \sin^2 \theta j & \sum_{j=0}^{K-1} \sin \theta j \cos \theta j \\ \sum_{j=0}^{K-1} \sin \theta j \cos \theta j & \sum_{j=0}^{K-1} \cos^2 \theta j \end{bmatrix} \begin{bmatrix} V_m \cos \phi_v \\ V_m \sin \phi_v \end{bmatrix} = \begin{bmatrix} \sum_{j=0}^{K-1} V_j \sin \theta j \\ \sum_{j=0}^{K-1} V_j \cos \theta j \end{bmatrix}$$

Further, show that for cycle with,
$\sum_{j=0}^{K-1} \sin \theta j \cos \theta j = 0$ and $\sum_{j=0}^{K-1} \sin^2 \theta j = \sum_{j=0}^{K-1} \cos^2 \theta j = \frac{K}{2}$.

Hence, derive a simple expression for calculating V_s and V_c. Compare and contrast with the full-cycle algorithm results.

Exercise 5

Evaluate the fundamental component of the square wave in Example-1 using the half-cycle Fourier algorithm. What conclusions do you draw?

Exercise 6

Suppose that the square wave in Example 1 also had a superposed DC component of 5 V, repeat exercise 5. Hence, refine your conclusions.

Exercise 7

One way to account for the decaying DC offset current during the estimation of the fundamental is to account for it in the signal model. Hence, consider the signal model to be $V(t) = V_m \sin(\omega_0 t + \phi) + V_0 e^{-t/\tau} + e(t)$. Assuming that the time constant 'τ' is known, develop a LS method to estimate V_m, ϕ and V_0. Compare the accuracy of this method with the full-cycle and half-cycle algorithms.

Exercise 8

Extend the full-cycle algorithm to measure the 3^{rd} and 5^{th} harmonics in a signal. Assume a suitable sampling frequency.

Thank You